

A MATHEMATICAL MODEL OF TONAL FUNCTION

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Abstract

I propose a mathematical model that formalizes the derivation of recent tonal harmonic theories that posit potential harmonic functions for each scale degree, such as Daniel Harrison's *Harmonic Function in Chromatic Music* and Ian Quinn's "Harmonic Function without Primary Triads". Mathematical groups that model tonal scale-degree functions will help to clarify the use of these functions to aid in composition and analysis. The mathematical representation of pitches is based on the ordered pair notation introduced by Alexander Brinkman (*Spectrum* 1986). Following an intuitive analytical discussion of the mathematical groups and the algebraic functions that relate them, I give examples of the useful distinctions among the tonal scale-degree functions that are clarified by this theory. I shall then use the distinctions among scale-degree functions supported by the mathematical model to reinforce and refine Ian Quinn's functional designations as well as to contribute to the current systems of part-writing techniques based on scale-degree functions.

Some of you may have been among the theorists who crammed into the crowded room in Boston last November where Ian Quinn was presenting his intriguing approach to theory pedagogy. While I was sitting on the floor in the front of that room, I found that he and I had been thinking along much the same lines with regard to the most economical and profitable ways of teaching harmony to undergraduates. Ian, however, was way ahead of me. His harmonic model, which emphasizes scale-degree function while simultaneously downplaying the role of primary triads as generators of those functions, is supported by his systematic use

of easily-remembered categories of scale-degree function and logical voice-leading patterns for each functional situation. His radical new analysis symbology,¹ which eschews both fundamental bass analysis and figured bass analysis in favor of a functional label plus a bass scale degree, elegantly makes explicit what I have been teaching as an implicit part of Roman-numeral analysis.

Figure 1 reproduces Quinn’s chart of functions, showing how each of the scale degrees may participate in a given chordal function—as a consonance, as a functional dissonance (aka essential dissonance), or as a non-functional dissonance (aka a non-chord tone). Within each group of consonant scale degrees, the two “central” consonances are what Quinn calls “triggers”. These are scale degrees 1 and 3 in a tonic chord, 4 and 6 in a subdominant chord, and 5 and 7 in a dominant chord. They correspond to Harrison’s “bases” and “agents”, respectively.² Functional dissonances, in Quinn’s model, may be treated in a suspension-like voice-leading pattern and must resolve down by step into a change of harmony, while non-functional dissonances must adhere to the traditional conventions concerning the use of non-harmonic tones. Quinn expands Harrison’s category of scale degrees called “associates” to include any diatonic degree that may complete a triad when combined with both of the “triggers”. In Figure 1, associates are the scale degrees on the periphery of (or sometimes just outside of) the consonance category for each function, namely scale degrees 5 and 6 in a tonic chord, scale degrees 1 and 2 in a subdominant chord, and scale degrees 2 and 3 in a dominant chord. The fact that some associates actually fall into the “functional dissonance” category does not alter their function as acoustical stabilizers. It simply means that they deserve special voice-leading treatment in terms of their resolution.

From this simple categorization of scale degree functions, Quinn generates an efficient set of voice-leading paradigms and procedures that greatly simplify the amount that a student will have to remember when completing a composition project. Further, his analytical system downplays harmonic distinctions that are less salient, such as the difference between IV, ii⁶, and ii₅⁶, thus streamlining the theory curriculum to free up time for more instrumental

¹This “functional-bass” analysis system was also independently developed by Charles Smith.

²The term “trigger” conveniently avoids confusion between the homophones “base” and “bass”.

composition and analysis of landmark works of tonal music.

My purpose here today is not simply to laud the accomplishments of Ian Quinn in his reworking of the sophomore theory program at Yale. Rather, I will be talking today about how simple mathematical techniques that I discovered in some obscure music theory research may be used to build a mathematical model of scale degree function.³ Through the exploration of the implications of this model, we shall discover that Quinn’s intuitively derived paradigms of scale-degree behavior are largely supported by my formalist groundwork. Further, my mathematical model supports Quinn’s theoretical system in several instances where it differs from previous scale-degree-based function theories. This will all be made clear shortly, but first I need to provide you with an intuitive understanding of how my mathematical model works.

Any tonal pitch or pitch class may be represented as an ordered triple of integers. We shall arbitrarily assign the ordered triple $(0, 0, 0)$ to the pitch C4, or “middle C”. The first component of the ordered triple represents a measurement of distance in semitones away from C4. Thus, the first integer in the ordered triple of the pitch D4 is 2, and F#3 is -6 . The second integer in the ordered triple represents a distance away from C4 measured in diatonic steps, without regard to chromatic inflection. Thus, the ordered triples for the pitches Db4, D#4, and D#4 all have 1 as their second component, though their first components are 1, 2, and 3, respectively. In other words, all “C”s have second component 0; all “D”s have second component 1; all “E”s, 2, and so on through “B”, which has second component 6. The first two components of the ordered triple thus completely encapsulate the information required to write a pitch in music notation, including how the pitch is spelled.⁴ The third component, then, adds more information about a pitch’s function than the diatonic spelling alone. It represents what part of a chord the given pitch fulfills, such as the root, third, or fifth. This is accomplished by counting the number of consonant skips within a chord that are required to arrive at the pitch. Thus, the third component in the ordered triple of the note C4 is 0, E4 is 1, and G4 is 2. The three components of the ordered triple may thus also be understood

³Fokker 1969 and Karp 1984.

⁴Brinkman (1986) uses this ordered pair notation for the computer representation of the diatonic spelling of a pitch.

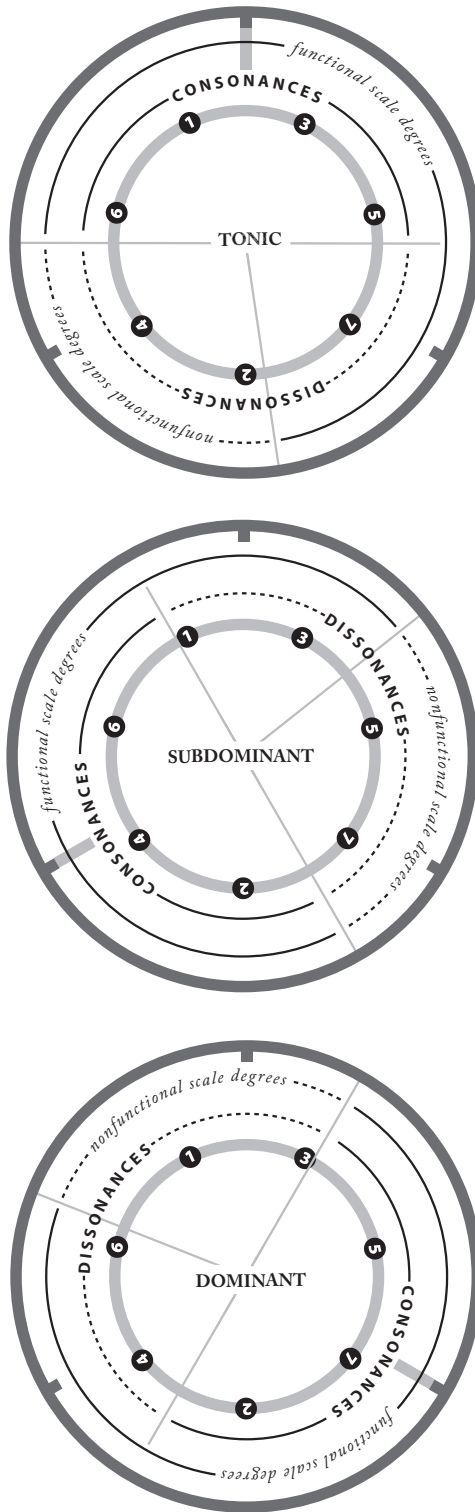


Figure 1: Quinn's Scale-Degree Functions (Quinn's Figure 6)

as step measurements in a 12-tone, 7-tone, and 3-tone division of the octave. We shall refer to the three components as a note's pitch number, step number, and arp number.

It is fairly simple to calculate a note's pitch number and step number using the information that I have just given you. It may still be useful, however, to have a mathematical formula that will generate the step number given the pitch number and the key in which the note is functioning. First, the variable a will stand for the pitch number or pitch-class number of the note for which we desire to derive an ordered triple. We shall also define a variable t to contain the ordered triple that represents the tonic pitch of the passage to be analyzed. Its three components (in left-to-right order) are t_1 , t_2 , and t_3 . Figure 2 gives the function ζ_t that maps integer pitch values to ordered-pair pitch-plus-step values within a key. The odd-looking brackets surrounding the fraction $\frac{a}{12}$ indicate that the value inside the brackets should be rounded down to the nearest integer less than or equal to $\frac{a}{12}$. The function ζ_t will spell any pitch in the chromatic scale properly relative to the key given by t . In other words, modal mixture, Neapolitan chords, and traditional augmented-sixth chords will all be spelled properly, but secondary dominants and diminished-seventh chords may not be spelled properly. For this reason, in any tonicization, no matter how brief, the analyst must derive a new t value for the key implied in the music.

$$\zeta_t(a) = \left(a, \frac{7 \cdot a - (((7 \cdot ((a - t_1) \bmod 12) + 5) \bmod 12) - 5) + 7 \cdot t_1 - 12 \cdot t_2}{12} \right)$$

$$y_t(a, b) = \left(a, b, (3 \cdot (-11 \cdot (a - t_1) + 19 \cdot (b - t_2)) + 5 \cdot (7 \cdot (a - t_1) - 12 \cdot (b - t_2))) - \left\lfloor \frac{7 \cdot (a - t_1) - 12 \cdot (b - t_2)}{5} \right\rfloor + t_3 \right).$$

Figure 2: Functions that map integer values for pitches to ordered pairs and triples

(On the screen you can watch this value being calculated for pitch class 1 in the key of B♭ major. Because the ordered pair for the tonic, B♭, is (10, 6), we shall use the value 10 in place of the variable t_1 , and the value 6 in place of the variable t_2 . Then the value 1 replaces the variable a . After solving the equation using a little bit of arithmetic, the result is the ordered pair (1, 1), meaning that the pitch class 1 is to be spelled as D♭ rather than C♯.)

Figure 2 also gives the function y_t that maps ordered pairs that contain a pitch number

and a step number to ordered triples that contain a pitch number, a step number, and an arp number. In this case, the variable b represents the step number (or step-class number) of the note being calculated. Note also that the odd-looking brackets surrounding the fraction invoke the nearest-integer function, where non-integers are rounded up or down depending upon their proximity to an integer.

(Let us watch the screen again to see the completion of the process of finding the ordered triple for the pitch class 1 in the key of B $\flat$ major. The first step in determining the ordered triple for each note in a piece of music will be to calculate the ordered triple of the tonic pitch using $(0, 0, 0)$ for the default t . In the interest of time, I shall not show the entire derivation of this value. As we already know, the function $\zeta_{(0,0,0)}(10)$ gives us a t value of $(10, 6)$. Further, $y_{(0,0,0)}(10, 6)$ returns a t value of $(10, 6, 2)$. Hence we replace t_1 in the function y_t with 10, t_2 with 6, and t_3 with 2. The variable a becomes the pitch number 1, and the variable b becomes the step number—also 1. Now we can calculate the arp number. The result is 0.)

As with the ζ function, the y function depends upon a tonal center to discriminate among potential tonal functions for a pitch. In this case, however, one may wish to use a temporary t value other than the prevailing tonic for reasons other than tonicization or modulation within the music. Take, for example, the ordered triples for a supertonic chord in the key of B $\flat$ major. As derived from the functions ζ and y , the ordered triples are $(0, 0, 0)$ for C4, $(3, 2, 0)$ for E $\flat$ 4, and $(7, 4, 1)$ for G4. We would expect any major or minor triad to have distinct arp-class numbers for root, third, and fifth, but this is not true of this particular triad. We must therefore measure some triads from a temporary tonic value other than the prevailing key as described in Table 1. If we use the method in Table 1 to determine the t values for both functions ζ and y , then major and minor chords will never be misspelled or display non-tertian arp numbers. In the case of the chord that we have just examined in the key of B $\flat$, the supertonic triad's closest chord member to scale degrees 1 and 5 is scale degree 4, E $\flat$. Using the E $\flat$'s ordered triple $(3, 2, 0)$ as a temporary tonic returns the ordered triples $(0, 0, -1)$ for C4, $(3, 2, 0)$ for E $\flat$ 4, and $(7, 4, 1)$ for G4.

It may be useful now to refer back to Figure 1 as we explore the use of ordered triples to support Quinn's scale-degree functions. Table 2 summarizes many of the possible scale-

Table 1: Cases in which a foreign tonic is to be used in calculating ordered triples

1. All tonicizations, no matter how brief, require a change in tonic reference pitch.
2. The tonic reference pitch for all non-dominant chords must be a chord tone other than the chordal seventh. Always use the chord tone closest to $\hat{1}$ and $\hat{5}$ on the infinite line of fifths.

degree functions and their ordered triple representations in the key of C. Those scale degrees that function simultaneously as the “trigger” of one category and an associate or functional dissonance in another category are only listed once in the table. The addition of such a scale degree to a chord of a different function will not change its ordered triple. For example, $\hat{4}$ has the same ordered triple when it is the root of a subdominant chord as when it is the seventh of a dominant chord. This is significant because $\hat{4}$ ’s sub/pre-dominant tendencies are not changed by its inclusion within a dominant-function harmony. (The same can be said of the lowered $\hat{6}$ when used as the seventh of a $\text{vii}^{\circ 7}$ chord, or $\hat{3}$ when used in a pre-dominant chord.) Contrary to Harrison’s view, but in accordance with Quinn’s tonal model, $\hat{2}$ is not in fact simply a dominant associate tacked onto a subdominant chord, but rather a true subdominant associate, helping to stabilize the subdominant “triggers” by completing a tertian triad with them. This is why $\hat{2}$ has two different arp numbers depending upon its function as dominant associate or subdominant associate. Except for $\hat{2}$, most associates and functional dissonances are thus borrowed from other functions, not just for acoustical stability, but also for a subtle “flavoring” of that other function. Table 3 summarizes the scale-degree functions implied by the third number in the ordered triple of a pitch. My mathematical model thus favors a distinction between the two types of functional “trigger” that Harrison calls “bases” and “agents”.

Making this distinction between “bases” and “agents” also allows us to devise very clear and concise doubling and voice-leading procedures for our students. In fact, this view of tonal function can be used to express doubling and voice-leading procedures far more succinctly than in most of the currently popular harmony textbooks. Figure 3 reproduces Quinn’s diagrams of voice-leading implications. I am fairly certain that Quinn does not make his

Table 2: Ordered Triples Representing Scale-Degree Functions in the Key of C

Function:	Tonic	Sub/Pre-Dominant	Dominant	Applied Dominant
$\hat{1}$ Do	(0, 0, 0)	—	—	—
$\sharp\hat{1}$ Di	—	—	—	(1, 0, 1) (V/ii)
$\flat\hat{2}$ Ra	—	(1, 1, 0)	—	—
$\hat{2}$ Re	—	(2, 1, 0)	(2, 1, 1)	(2, 1, 1) (V/V)
$\flat\hat{3}$ Me	(3, 2, 1)	—	—	—
$\hat{3}$ Mi	(4, 2, 1)	—	—	(4, 2, 2) (V/ii)
$\hat{4}$ Fa	—	(5, 3, 1)	—	(5, 3, 2) (V/III)
$\sharp\hat{4}$ Fi	—	—	—	(6, 3, 2) (V/V)
$\hat{5}$ Sol	—	—	(7, 4, 2)	—
$\flat\hat{6}$ Le	—	(8, 5, 2)	—	—
$\hat{6}$ La	—	(9, 5, 2)	—	(9, 5, 3) (V/V)
$\flat\hat{7}$ Te	—	(10, 6, 2) (V7/IV)	—	(10, 6, 3) (V/III)
$\hat{7}$ Ti	—	—	(11, 6, 3)	—

Table 3: Scale-Degree Functions Implied by the Arp Numbers in the Key of C

Arp Number	Tonic	Subdominant	Dominant
... -3, 0, 3, 6 ...	Base	Associate	Agent
... -2, 1, 4, 7 ...	Agent	Base	Associate
... -1, 2, 5, 8 ...	Associate	Agent	Base

students study these complex charts in order to learn how to write well-formed tonal progressions. I presume that Quinn, like many of us, has devised some simple and memorable ways of teaching students about where the different scale degrees like to move. While the complexity of Figure 3 parallels the sophistication of tonal practice in the 18th and 19th centuries, these charts may not necessarily be a valuable resource in the theory classroom. I believe that we can offer students a much simpler set of procedures that they have a fighting chance of remembering, and that neatly encapsulate the most useful techniques applicable to nearly all voice-leading situations.

First, Table 4 addresses the natural flow of tonality, from tonic function, through pre-dominant function, to the cadence points at the end of each phrase. Chords that violate this flow must be part of a passing or neighboring progression or a typical cadence pattern. Next, Table 5 gives a checklist to aid in determining the doubling of any chord. One point

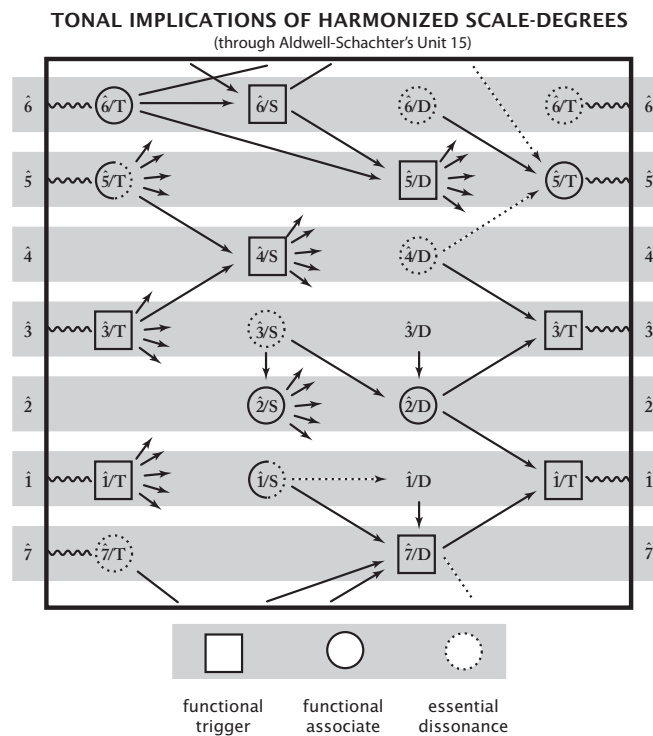
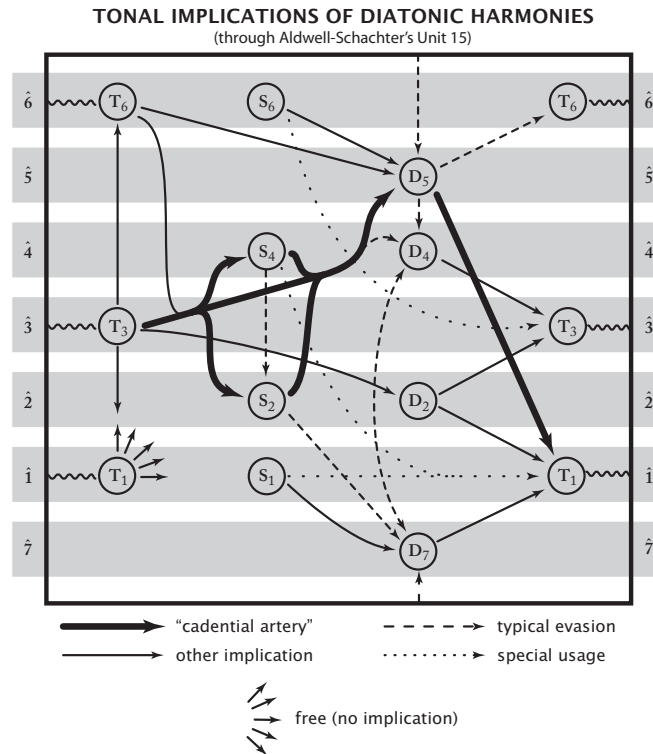


Figure 3: Voice-Leading Implication Chart (Quinn's Figure 3)

to emphasize with the last item in this list is that one should prefer to double the base of the current harmonic function only, passing over any bases of other categories that may be acting as associates of the current function. Table 6 offers simple voice-leading techniques for handling each category of scale degree, including non-chord tones, essential dissonances, agents, and a catch-all procedure for bases and the remaining associates. Avoiding parallel fifths and octaves, satisfying the doubling requirements, and following these voice-leading procedures sufficiently restricts the possibilities so that students will not be lost regarding what to do with each voice in a chorale-style composition.

Table 4: Cases when retrogressive motion is allowed

1. Passing and neighboring chords
2. Plagal, deceptive, and half cadences

The treatment of agents is written in broad terms as a procedure in Table 6, but may be made explicit as seen in Figure 4. In this diagram, the ‘P’ or ‘R’ accompanying each arrow specifies the type of motion, progressive or retrogressive, for which the arrow gives the normative voice resolution. In general, progressive motion is clockwise around the diagram, and retrogressive motion is counterclockwise.

Table 5: Doubling Procedures

1. Do not double leading tones, dissonances, or inflected scale degrees.
2. Double the bass of passing and neighboring chords of the sixth, including all $\frac{6}{4}$ chords.
3. Otherwise, prefer to double bases when possible and chord roots when not.

As students progress, a teacher may wish to relax the restrictions on the resolution of agents to allow agents to skip by third in the opposite direction when they appear in an inner voice. Further, a teacher may also wish to discuss the few situations in which melodic necessity trumps the given voice-leading procedures, such as when $\hat{6}$ resolves up to $\hat{7}$.

Although it is not directly related to my mathematical model, allow me now to talk briefly about Quinn’s analysis system. In functional-bass analysis, every chord gets a functional la-

Table 6: Voice-Leading Procedures

1. Non-functional dissonances must follow traditional non-chord-tone patterns.
2. Functional dissonances resolve down by step when the music achieves the next function.
3. Agents resolve by step to the nearest base (progressive) or associate (retrogressive) when the music moves to another function.
4. Inflected bases and associates resolve by minor second in the direction of their inflection.
5. Otherwise, try to avoid leaps by sixth or seventh, generally preferring stepwise motion over leaps.

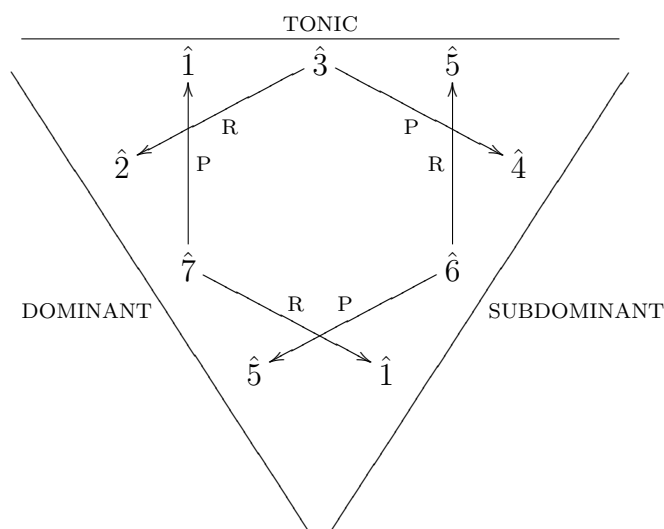


Figure 4: Chart of Agent Resolutions

bel, ‘T’, ‘S’, or ‘D’, and a bass scale-degree number. If you happen to use moveable-do solfège in your ear-training classes, it may be useful to modify the system slightly to use solfège syllables in place of scale degrees for the bass notes. It is in the use of analytical symbols to demonstrate one’s understanding of a passage of music that our scale-degree-based model of function really shines. Not only can one label every chord according to its function within the progression, but one may also, when performing a detailed analysis, label the role that every note plays within the music. While this is perhaps too tedious an exercise for every assignment, another labelling practice is perhaps even more useful. Every agent, dissonance, and inflected base or associate has an expected direction of resolution, and an arrow may

be drawn from each tendency tone to its resolution tone. (On the screen (Figure 5) you can see an example of such an analysis. Arrows point out the resolution of every non-functional dissonance, functional dissonance, agent, and inflected base or associate. Non-functional dissonances include the accented passing tones above the cadential $\hat{6}_4$ in the final measure; an inflected associate may be found in the Neapolitan chord in the penultimate measure; and an inflected base may be found in the French augmented-sixth chord in the antepenultimate measure. The only unusual resolutions in this example are part of the passing and neighboring dominant chords near the beginning. These atypical resolutions would make excellent fodder for classroom discussion of passing and neighboring chords.) As this example shows, the habit of marking stepwise resolutions will serve as a reminder for students when doing a composition or arranging assignment and will also offer the opportunity for students to observe the ways in which other composers evade these tendencies. The example on the screen also gives a more detailed analysis of the harmony than functional-bass symbols alone give, by adding figured bass to distinguish among the possibilities for each function/bass combination. In other words, we may distinguish among IV, ii^6 , and ii_5^6 by calling them $S4_3^5$, $S4^6$, and $S4_5^6$, respectively. (The example on the screen, in fact, includes an $S4_{b3}^6$, aka the Neapolitan.)

I hope that my mathematical model has lent some credence to this scale-degree-based functional theory. Further, I encourage you to consider incorporating some aspects of this emerging pedagogy of tonal harmonic practice into your own teaching. I do not, of course, advocate using my equations and ordered triples as tools in the undergraduate theory classroom. The concepts that the math models, however, are indeed pedagogically useful. It is my belief that the students in my classes who—even after my frequent reminders—forget to treat the leading tone differently from other scale degrees do so because they are still more concerned with roots, thirds, and fifths, than harmonic function. If we turn our priorities around completely, however, and ask students to think about scale degrees first, and about triads and seventh chords last, I believe that even the students who continue to forget to add an accidental to $\hat{7}$ in minor will nevertheless gain a richer understanding of tonal harmony and musical composition.

BM: D7 T1 T3 D4 T_♭3 D2 T1 D4/IV S6 S_♭6 D5 D7/vi S6 S4 D5 D5 T1

Figure 5: Example of a Complete Functional-Bass Analysis

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