Charting Enharmonicism on the Just Intonation *Tonnetz*: A Practical Approach to Neo-Riemannian Analysis

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Spring, 2002 - 2003

Abstract

In this study, I propose a theory of voice leading for use with the tonal space of the just-intonation *Tonnetz*. The theory is based largely upon established theories of diatonic voice leading and chromatic harmony, and also draws upon the principles of just intonation within the tonal system. The unambiguous expression of intervals as untempered ratios is dependent upon an established seven-step diatonic scale in which the intervals can function. Tuning corrections must be made at specified junctures within the just-intonation diatonic system in order to show its stability in comparison with chromatic harmony in just intonation. Only then can the various types of enharmonic progressions be unambiguously charted spatially as directed motion on the *Tonnetz*. In addition to providing a diatonic context for parsimonious voice leading between any two tertian sonorities, the just-intonation *Tonnetz* model allows for seventh chords, non-chord tones, augmented-sixth chords, other chromatically inflected chords, and extended tertian chords to be expressed on the tonal network along with the traditional triadic transformations. Examples of the analytic method are given here through the graphing of several typical enharmonic progressions drawn from the two Rhapsodies, Op. 79, of Johannes Brahms. Finally, a case-by-case method for treating atypical resolutions of normally functional chords is suggested through a just-intonation analysis of the opening of the Prelude to Wagner's *Tristan und Isolde*.

Introduction

In analyzing chromatic music as voice leading transformations, one is likely to see a diversity of chord types and qualities, most with tonal functional expectations, featuring both traditional and non-traditional resolutions. In recent research, certain transformations have been selected by neo-Riemannian theorists as displaying a special "parsimonious" status.[1] These studies, however, treat only a small fraction of all of the chord successions seen in chromatic music. I propose that a definition of a kind of diatonic parsimony will also allow for a worthwhile practical view of music as a progression of closely-related harmonies through tonal space.[2] In the first section of this paper, I shall outline the diatonic theories that support the reasoning behind the just-intonation *Tonnetz* model given in part two. Part three presents examples of the analytical method taken from the two Rhapsodies, Op. 79 of Johannes Brahms. In part four, a case-by-case method for treating atypical resolutions of normally functional chords is suggested through an analysis of the opening of the Prelude to Wagner's *Tristan und Isolde*.

[1] For a background and introduction to neo-Riemannian theory, see Hyer 1995, Cohn 1997, and Morris 1998. For a series of articles on the subject and a bibliography, see the Neo-Riemannian Theory issue of the *Journal of Music Theory* (42/2, Fall 1998).

[2] This kind of parsimony is not restricted to transformations that preserve common tones. Instead, exclusively stepwise (and common-tone) motion is the constraint for "diatonic parsimony." These voice-leading restrictions are given by Agmon 1991 and several other studies of diatonic voice leading.

I. The Diatonic Basis for Chromatic Music

A summary of certain recent theories of diatonic voice leading will establish the basis for such a diatonic understanding of all voice leading.[3] Agmon (1991) provides a mathematical description of mod-7 diatonicism that privileges triadic tonality when stepwise voice leading constraints are in place. Agmon's system accomplishes this through induction of the necessity for minimal stepwise or common-tone voice-leading connections between three- and four-note cyclically-generated sonorities. In other words, among triads and seventh chords there are always stepwise and/or common-tone voice motions that are surjective (onto) and do not exceed the number of notes in the larger chord. (This "efficiency constraint" is a kind of modified injective (one-to-one) relation to allow for all possibilities of transformations between both three- and four-note chords.) Agmon's proof demonstrates that in a seven-scale-step diatonic universe, triads and seventh chords are the only sonorities that satisfy this condition of efficient linear connections.

For analytic purposes, the significance of this finding lies more in the efficiency constraint itself than the demonstration of the power of tertian harmony. Agmon shows that any three- and/or four-voice tertian-chord transformation is diatonically parsimonious.[4] The neo-Riemannian theorists[5] focus narrowly on the restricted parsimony of the contextual inversions

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[3] The diatonic basis of chromatic harmony has been extensively debated in the literature. Proctor (1978, 149ff) and McCreless (1983, 60-62) assert that when chromatic progressions transcend the established functional harmonic paradigms (diatonic root motion, diatonic resolution of unstable intervals, etc.), a diatonic harmonic foundation is no longer present. On the other hand, Smith (1986, 109) and Harrison (1994) give arguments for the diatonic basis of harmonic function in chromaticism, and Brown (1986) presents Schenker's chart of Stufen (containing every chromatic degree except ♯IV/♭V) as a counterexample to McCreless and Proctor. Jones (2002, 111) argues for the local (but not global) diatonic interpretation of chordal successions, and builds upon Smith's view, asserting that a fundamentally diatonic distinction between "stepwise motion" and "chromatic inflection" is part of every listener's perception of chromatic music. My own work is based largely upon this understanding of chromaticism.

[4] See note 2.

[5] See note 1.

(*Wechsel*) that preserve one or two common tones,[6] without considering within their theory the transposition operations (*Schritte*) at play in most chromatic music.[7] Klumpenhouwer (1994) catalogues all 24 triadic transformations, which can all be shown to be diatonically parsimonious through Agmon's (1991) proof. (They all can be expressed as having maximally close diatonic voice-motions that satisfy Agmon's conditions.) A transformational labelling system would thus be possible as in Figure 1 so that a diatonically parsimonious connection could be specified between any two triads.[8]

Seventh chords could be incorporated into Klumpenhouwer's neo-Riemannian transformational model, but several problems must be overcome. Major-major seventh chords, 4–20 (0158), major-minor seventh chords, 4–27i (0269), and minor-minor seventh chords, 4–26 (0358), could be treated in the same manner as the triads built on the same root. This is similar to the manner in which Hanford (1987) handles seventh chords in her voice-leading models. Hanford simply adds the seventh into whatever other voice would typically resolve to the pitch class one step below the seventh. Treating the sevenths as appended onto to a triadic voice-leading model implies a perspective on chordal sevenths not unlike Schenker's view that the triad is the primary voice-leading entity, and the seventh is actually an elided 8 – 7 passing motion above the root.[9] Agmon (1997, 4:15) succinctly explains Schenker's ideas about the "essential" seventh: "The seventh chord of free composition is a passing-tone formation ... of a very special type, unfamiliar to strict counterpoint."

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[6] Morris' (1998) "obverse" transformations preserve one common tone.

[7] As any *Schritt* can be expressed as the combination of two *Wechsel*, it is possible to express *Schritt* operations as functions of neo-Riemannian transformational nomenclature. These combinations of inversional relations, however, do not address the efficient voice leading connections involved.

[8] All *Tonnetze* given in this paper will use pitch class numbers, rather than note names, and the neutral letters X and Y will be used to represent pitch classes 10 and 11. Even though my theory supports the expression of enharmonic shifts by respelling note names, it is essential to recognize the enharmonic equivalence between differently-spelled pitch classes to attain an overall perspective of a piece, even when equivalent pitch classes are tuned differently.

[9] See Clark 1982, Agmon 1997.

Figure 1a *Schritte* **and** *Wechsel* **from C major and C minor**

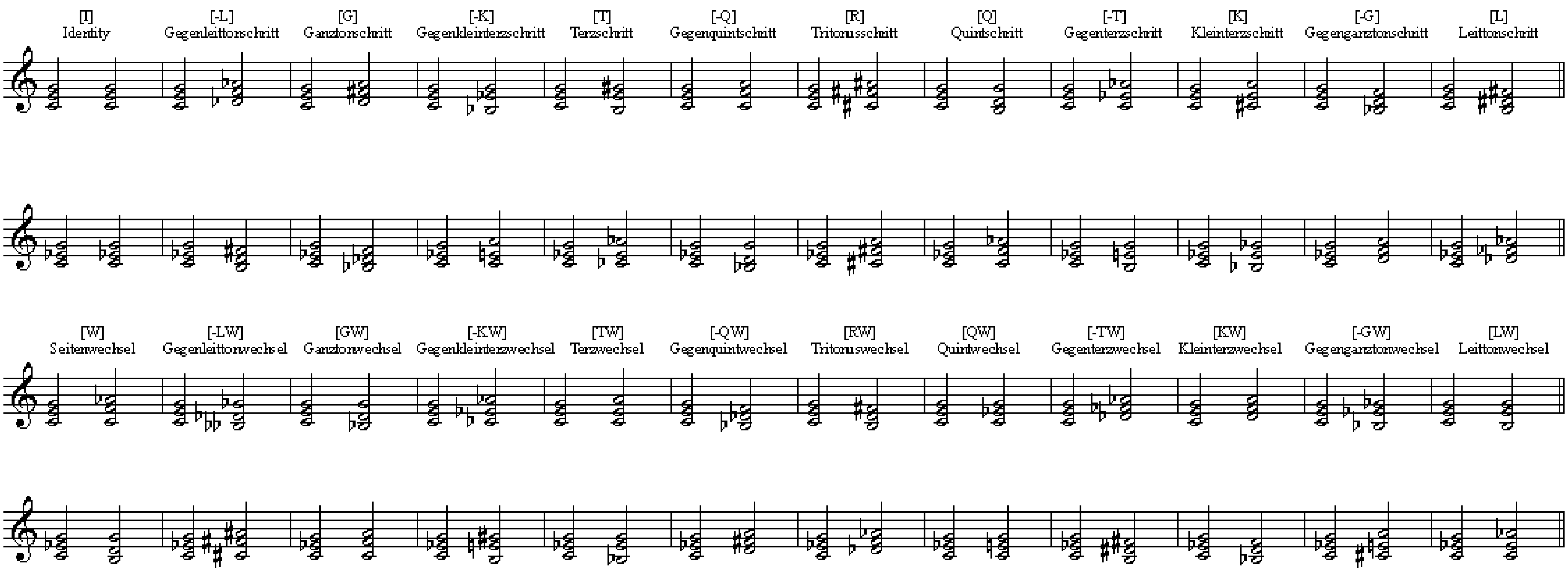

Figure 1b Klumpenhouwer's 24 transformations on the *Tonnetz* from C major and C minor.

Schritte and *Wechsel* relative to C major (C,+)

```
2   5---8---Y---2---5
    |-L/|T /|Q /|-G/|
    | / | / | / | / |
    |-TW|/LW|/GW|-QW|
X---1---4---7---X---1
|R /|K /|I /|-K/|
| / | / | / | / |
|-LW|/TW|/QW|-GW|
6---9---0---3---6   9
|G /|-Q/|-T/|L /|
| / | / | / | / |
|/KW|/ W|-KW|/RW|
2---5---8---Y---2   5
```

Schritte and *Wechsel* relative to C minor (g,-)

```
2   5---8---Y---2---5
    |RW/|-KW|SW/|KW/|
    | / | / | / | / |
    |/ L|/-T|/-Q|/ G|
X---1---4---7---X---1
|-LW|-GW|QW/|TW/|
| / | / | / | / |
|/ R|/-K|/ I|/ K|
6---9---0---3---6   9
|-QW|GW/|LW/|-TW|
| / | / | / | / |
|/-G|/ Q|/ T|/-L|
2---5---8---Y---2   5
```

The seventh may also be viewed as an extension of the triadic expression of the overtone series of the fundamental bass. Adding the seventh partial, which many theorists discard as being "out of tune" relative to most traditional tempered scales, onto the first six partials produces a tuning of the major-minor seventh chord with no beats.[10] Reversal of the position of the major and minor thirds within the triad will give a minor-minor seventh chord. The extended overtone series view argues against Schenker's rejection of the "essential" seventh as being "unnatural" (an invention of free composition). It nevertheless supports a similar conclusion about the status of the essential seventh as an extension within a harmonic theory dominated by the triad.

With the inclusion of chordal sevenths, these neo-Riemannian transformations are very powerful with regard to major and minor chord successions, but Klumpenhouwer's transformations do not include diminished and augmented chords. Because neither diminished nor augmented triads belong to set class 3–11 (037), no transformation of major or minor triads into augmented or diminished triads or vice versa can be expressed as either a transposition (*Schritt*) or a contextual inversion (*Wechsel*). It may be possible for augmented and diminished triads to be interpreted as simply chromatic inflections of major or minor triads (with a specific contrapuntal resolution implied by the chromatic passing motions). Likewise, half-diminished seventh chords, 4–27 (0258), and fully diminished seventh chords, 4–28 (0369), would be inflected minor-minor seventh chords, 4–26 (0358), and major-minor seventh chords, 4–27i (0269). Unusual resolutions of these sonorities, however, and the use of augmented-sixth chords

[10] This tuning violates the sanctity of Zarlino's *Senario*, discussed in Walker 1996. Walker limits his just-intonation model to the sixth overtone as well, calling the *Senario* the "5-limit system" (indicating that 5 is the largest prime number that generates interval ratios in the system). Regener 1975 demonstrates that the "overtone" dominant seventh tuning is an augmented sixth chord, and that the major-minor seventh uses a 10:7 tritone. As will be seen below, the *Tonnetz* imposes a 5-limit restriction on the intervals used.

present further complications of this matter with regard to transformational interpretations of chord successions in practical tonal contexts.[11] No consistent neo-Riemannian transformational methodology may yet be available to describe relations among all possible tonal chords. Nevertheless, Agmon's (1991) linear transformations ensure a diatonically parsimonious voice leading connection between any pair of three- or four-note tertian sonorities, as long as they are spelled diatonically.[12] A *Tonnetz* representation of any of these transformations should therefore still have meaningful implications with regard to voice leading and tonal relationships.

The attempt to represent the various types of chords on the *Tonnetz* presents a new dilemma related to diatonic parsimony and tonal function. While major and minor triads will be represented as triangles, and diminished and augmented triads as straight lines, seventh chords present more complicated shapes. For instance, Figure 2 shows the representation of a B diminished seventh chord (Y258) on the *Tonnetz*. If the *Tonnetz* is representing twelve-tone equal temperament, the flat chart actually wraps around upon itself into torus, as seen in Figure 8.[13] Because the network wraps around so that there is only one instance of any pitch class, the question of which pitch classes represent the chord members on the flat version of the chart becomes arbitrary. Any position on the network labelled with pitch class number 11, 2, 5, or 8 will suffice for representing this sonority, and, in fact, it would be accurate to connect all 11s, 2s, 5s, and 8s with an infinite straight line. This fact resonates with the idea that tonal composers

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[11] Several articles in the Neo-Riemannian Theory issue of the *Journal of Music Theory* (42/2, Fall 1998) explain seventh chord transformations in twelve-tone equal temperament, including Childs, Douthett/Steinbach, and Gollin. The focus in these articles is primarily on the inversion-related 4-27 (0258) dominant/half-diminished seventh pair. Douthett and Steinbach extensively discuss the chord progression at the opening of Wagner's *Tristan und Isolde*, for which I will offer an interpretation in just intonation in part four.

[12] A labelling system for all of the diatonically parsimonious transformations between two tertian harmonies may prove to be useful, but is beyond the scope of this study.

[13] The assertion in Cohn 1997 that the equal-temperament *Tonnetz* is a *hyper*torus comes from Hyer 1989. Hyer, for some reason, did not consider tempering only major and minor thirds sufficient for creating equal temperament.

often enharmonically reinterpret the sonority and resolve it to one of many other more stable chords or simply to another diminished seventh chord.

Figure 2 B diminished seventh on the *Tonnetz* (X=10, Y=11)

```
4   7   X   1   4   7   X   1

0   3   6   9   0   3   6   9

8   Y--2--5--8   Y   2   5

4   7   X   1   4   7   X   1

0   3   6   9   0   3   6   9

8   Y   2   5   8   Y   2   5
```

If, for example, the B diminished seventh chord resolves enharmonically as an F diminished seventh chord to a G♭ minor chord (6X1), it would be represented exactly the same way as if it were reinterpreted as an E♯ diminished seventh resolving to F♯ minor. Notwithstanding the enharmonic distinction between these two chords, they would be represented as being precisely the same position on the equally-tempered *Tonnetz*. Because these are two conceptually different entities in diatonic tonal theory, one could distinguish between them spatially by choosing two different instances of the triad (691) on the flat representation of the *Tonnetz*. The choice of location, however, is problematic because any pitch class 5 resolving to any pitch class 6 on the equally-tempered *Tonnetz* is equivalent. Many neo-Riemannian theorists require motion to the closest instance of the second pitch class.[14] This requirement has no meaning without expressible harmonic function on the equally-tempered

[14] Harrison 2000 takes this stance, using a just-intonation model to represent our perception of motion in any tonal space, including the equally-tempered *Tonnetz*.

Tonnetz, since one pitch class on the equally-tempered *Tonnetz* can move by the same tonal function to *any* instance of its resolution pitch class on the network. Twelve-tone equal temperament thus removes any established spatial representation of the tonal function of the various members of the diminished seventh chord, or any tertian chord, and thus fails to represent diatonic parsimony spatially.

Because of the functional nature of all tonal voice-leading, the necessity of a diatonic interpretation for tertian harmonies comes as an integral part of the assumption of tonality, including the tonality of chromatic harmony. If the *Tonnetz* is to be used to chart paths through a spatial model of tonality, it must be able to distinguish between two enharmonically equivalent key areas unequivocally as to their location. The *Tonnetz* can be a very powerful tool for showing progressions spatially, but only if there can be only one representation of any given progression based on the prevailing tonal situation. A tonal space that represents functional intervals consistently and distinguishes between different interval spellings will thus be necessary for the representation of harmonies with tonal implications.

Figure 3 The intervals of the just-intonation *Tonnetz*

```
1   4   7   X---1   4   7   X
|            5:6
|4:5
|
9   0   3   6   9   0   3   6
       /             \
      |               \24:25
      |                \
5   8 | Y   2   5   8   Y   2
      |            /
      |15:16      /2:3
     /           /
1   4   7   X   1   4   7   X
```

II. Diatonic and Chromatic Harmony on the Just Intonation *Tonnetz*

If we are to use a just-intonation flat *Tonnetz*, there are several new obstacles to overcome.[15] The frequency ratio between members of each of the interval cycles represented on the *Tonnetz* will be fixed as seen in Figure 3. The minor third will be a 6:5 ratio, the major third will be a 5:4 ratio, the perfect fifth will be a 3:2 ratio, and so on. The two different tunings of the half step, 15:16 and 24:25, distinguish between the minor second and the augmented unison, respectively.

Figure 4 shows a progression of the chords I-vi-ii-V-I in C major on the just-intonation *Tonnetz*. The interval between the first C and the last C is 80:81. This syntonic comma between the $\hat{2}$ of the dominant chord, with its ratio of 9:8 relative to tonic, and the $\hat{2}$ of the supertonic chord, with its ratio of 10:9 relative to tonic, presents the first obstacle to the stability of diatonic progressions.[16] To solve this problem, one can simply introduce a temperament system. A mean tone tuning would suffice for this purpose, but would ruin the cycles of simple intervals of the just-intonation *Tonnetz*. Perhaps the simplest solution to this problem would be to embrace the imperfections of just intonation and allow these simple diatonic progressions to move to a tonic that is 80:81 relative to the original, and thus a different location on the *Tonnetz*, as seen in Figure 4. Such a solution, however, must dismiss the listener's memory of an accurately-tuned tonic pitch. In fact, the expected pitch relationship of the dominant chord's (or possibly some

[15] Using a just tempered model to represent music for instruments tuned using a tempered tuning system is hardly contradictory. It remains consistent because a temperament is a set of compromises made in order to create an instrument that is both usable in many keys and a reasonable approximation of just intonation. For a description of temperament as a compromise relative to just intonation, see Isacoff 2001, 97-104, and Walker 1996, II.6

[16] Riemann ([1914-15] 1992, 100) himself treats this problem, stating that the "*enharmonic identification* of acoustical values that differ by a syntonic comma is simply indispensable to our musical hearing." (Hyer 1995, 106, and Harrison 2000, 22)

other chord's) root to the original tonic will be audibly frustrated every time this comma slip occurs.

It would thus be advisable to find a solution that minimizes the number of comma discrepancies within the system. Focusing on the $\hat{2}$ disparity accomplishes this comma minimization. The only comma shift in the tuning system will thus be a retuning of the common tone $\hat{2}$ from 10:9 to 9:8. This retuning will occur in any ii-V-I progression that is approached by vi or IV, in any ii^{7}-V-I progression that is approached by I, vi, or IV, and in any French augmented-sixth chord resolution that is approached by I, vi, iv, or ii^{7}.[17] It will be worthwhile now to follow through with the implications of this compromise in order to finish outlining the just-intonation model that renders all diatonic progressions stable on the *Tonnetz*.

Figure 4 C major I-vi-ii-V-I on the just-intonation *Tonnetz*, showing the 80:81 comma

```
1   4   X   1   4--7
               /|I/
           vi/ |/
0   3   6   9--0   3
           /|
       ii/ |
8   Y--2--5   8
    |V/
    |/
4--7   X   1   4
|I/
|/
0   3   6
```

[17] Walker (1996) suggests the same solution, but does not treat seventh chords specifically.

In order to have a *Tonnetz* that shows chromatic and enharmonic progressions as motion on the *Tonnetz* but diatonic progressions as stable around a single tonic triad, we can use a "tube" *Tonnetz*. Figure 8 shows that, rather than tempering all intervals into a twelve-tone equal temperament torus *Tonnetz*, the comma correction would join only two edges of the network into a cylindrical shape. This tube model would allow I-vi-ii-V-I, I-IV-ii-V-I, and similar progressions to loop around the tube back to the original tonic through one 81:80 comma correction. Any other progression by cycles of perfect fifths, major or minor thirds, semitones, or tritones, however, would travel down the tube in one direction, cycling around the surface at a particular rate, but never returning to the original pitch class.[18] It is important to remember that this joining of edges into a tube is not representative of a tempered interval system, but rather a result of a comma shift imposed on certain progressions. The location of the juncture on the tube *Tonnetz* will thus be different in the various progressions according to where the tuning adjustment is made. Further implications of this model will be explored further shortly. First, it will be necessary approach the second obstacle in the just-intonation system: the tuning of seventh chords using exclusively ratios present on the *Tonnetz*.

As discussed above, the ideal dominant seventh chord (V^7) has a 5:4 6:5 25:21 tuning.[19] 7-limit ratios are not represented on the just-intonation *Tonnetz*. The simplest solution would be to use the version in Figure 5a, which is 5:4 6:5 6:5. The seventh in this case resolves to $\hat{3}$ by 25:27.

[18] Cohn (1996, 11) uses the term "vertigo" to describe the effect of such progressions, "which at once divide their space equally and unequally.... The enharmonic shift cannot be located: it occurs everywhere, and it occurs nowhere." Cohn's analogy resonates with the spatial imagery of progressions spiralling down an endless tube.

[19] Here I use ratios for each interval in the sonority from bottom to top so that chords comprised of more complicated intervals can be introduced later without confusion.

An even better approximation of the 5:4 6:5 25:21 tuning occurs if no comma correction is applied to the subdominant chord's $\hat{4}$ as it is held into the V^7 as a common tone. This amounts to correcting the syntonic comma within the chord itself, and the resulting tuning is 5:4 6:5 32:27. In this case, $\hat{4}$ will resolve to $\hat{3}$ by 15:16, which is a just-intonation minor second. As the subdominant $\hat{4}$ is not contiguous with the rest of the chord on the *Tonnetz*, for simplicity and clarity, the geometry of the tube *Tonnetz* can be used to represent the chord entirely on the dominant side of C major. The result is exactly the same as the representation in Figure 5a.

The ideal minor-minor seventh chord (ii) is a 6:5 5:4 25:21 tuning. This chord must similarly be simplified so that it contains only intervals of 5:4 and 6:5. This results in a 6:5 5:4 6:5 tuning. Because this chord is build on $\hat{2}$ at 10:9 above tonic in order to hold the tonic pitch as a common tone, the syntonic-comma correction must occur following this chord in a progression (except for $\hat{4}$ as discussed above).[20] It is also possible here to correct the comma within the chord, so that the third, fifth, and seventh come from the subdominant side and the root of the chord comes from the dominant side. The tuning here will be 32:27 5:4 6:5, and the seventh will resolve to $\hat{7}$ by 15:16. This allows the comma to be corrected within the V^7 chord as well, with no common tone retunings between any of the chords. Two equivalent C major *Tonnetz* representations of this chord are shown in Figure 5b.

One possible ideal half-diminished seventh chord ($ii^{ø7}$) is a 6:5 7:6 9:7 tuning. Here the tritone (7:5) will not beat. (In fact, the chord as a whole will have no beats, as all chord members belong to the overtone series with a fundamental 1:5 below the chord's root.)[21] The simplest 5-limit compromise is a 6:5 6:5 5:4 tuning. It is important to notice that the seventh in both tunings

[20] In my personal experimentation I have found this comma shift to be acceptable to my ear. Several other scholars have argued along the same lines. Further research is required to justify the perceptual acceptability this shift.

is the same (3:2 above the third), but in this practical tuning the fifth is slightly higher relative to the other chord members than it was in the ideal tuning. Once again the chord must be built on 10:9 above tonic for the seventh to be a correctly-tuned $\hat{1}$. When the comma is corrected after this chord and before the next chord, $\flat\hat{6}$ will resolve to $\hat{5}$ by 15:16 (as opposed to 25:27 when the root is 9:8 above tonic). The comma can also be corrected within the chord in this case. As with the minor-minor ii^7, only the root ($\hat{2}$) will be taken from the dominant side, and the tuning will be 32:27 6:5 5:4. Both of these tunings also suffice for leading tone half-diminished seventh chords ($\text{vii}^{\o 7}$). The two equivalent *Tonnetz* representations of this chord are given in Figure 5c.

The "purest" fully diminished seventh ($\text{vii}^{\circ 7}$) chord is in fact a compromise to begin with. The cleanest compromise is a 6:5 25:21 6:5 tuning. Here, both diminished fifths are 10:7. The augmented second is 7:6. In order to approach an ideal tuning within 5-limit just intonation, and also to preserve functional resolutions, I propose that the comma always be corrected within the chord in this case. The root and third of the chord will be drawn from the dominant chord, and thus resolve to $\hat{1}$ by 16:15 and 9:8 as they should ideally. The fifth and seventh will be drawn from the minor subdominant chord, and will resolve to $\hat{3}$ and $\hat{5}$ both by 15:16. According to this explanation, the chord should be tuned 6:5 32:27 6:5, and be represented in C major on the *Tonnetz* as shown in Figure 5d. Notice the path by which each chord member resolves to the appropriate member of the tonic chord as compared to the resolutions in typical authentic and plagal motions. This interpretation of the chord represents the view of diminished sevenths proposed by Harrison 1994. For simplicity and clarity, however, the tube *Tonnetz* can be used to represent the chord entirely on the dominant side of C major as seen in Figure 5e.

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[21] It should be noted that, because of the just-intonation intervals that make up this chord, the half-diminished seventh is not an inversion of the major-minor seventh chord in just intonation.

The Italian and German augmented-sixth chords (4:5:6:7) function similarly to a secondary diminished seventh chord (vii$^{\circ 6}_{5}$/V), except that $\hat{6}$ becomes $\flat\hat{6}$. They will thus also split into dominant and subdominant tunings within the chord. In this case, only the $\sharp\hat{4}$ will be drawn from the dominant side, as the rest of the German sixth chord should be a consonant major triad. With this tuning, all chord members resolve by diatonic minor second (15:16 or 16:15) to members of the V chord. If the chord resolves instead to the major cadential $^{6}_{4}$, $\flat\hat{3}$ resolves to $\natural\hat{3}$ by 25:24. The only reason to respell $\flat\hat{3}$ as $\sharp\hat{2}$ is to aid in reading the notation, because the perfect fifth, when tuned as a doubly-augmented fourth, would become highly dissonant (625:432 rather than 3:2). As with the dominant- and diminished-seventh chords, the chord most practically should be consolidated (via the tube wrapping) into a single location on the *Tonnetz*. The $\sharp\hat{4}$ that is used here, however, is still not completely contiguous with the rest of the chord, as can be seen in Figure 5f.

The French augmented-sixth chord, as seen in Figure 5g, contains $\hat{2}$, which resolves by common tone into the dominant chord. The distribution here is thus once again two notes on the dominant side and two notes on the subdominant side. The tuning of this chord (ascending from the bass $\flat\hat{6}$) is thus 5:4 9:8 5:4. After resolving the *Tonnetz* notation to one side or the other, the construction on the *Tonnetz* is still rather strange-looking, but will suffice, as seen in Figure 5g.

Now that the tuning system has been established, it will be possible to explore the just-intonation interval cycles on the tonal network. These can be found in Figure 6. Because of the tube wrapping of the *Tonnetz,* certain interval cycles will now intersect periodically. For example, a descending major third cycle from a C major triad (CM, A♭M, F♭M, D♭♭M, etc.) will intersect an ascending minor third cycle from a C major triad (CM, E♭M, G♭M, B♭♭M, D♭♭M, F♭♭M, etc.) at D♭♭ major. This intersection makes sense because the two D♭♭ major chords are diatonically spelled the same (though originally different in tuning by a syntonic comma on the flat *Tonnetz*). If this tube *Tonnetz* were instead to cause an intersection of two enharmonically equivalent but diatonically *different* chords, then it would fail to represent chromatic harmony properly within diatonic space. Because the various interval cycles wrap around the tube at different rates, it may be necessary to depict this wrapping as jumping from the bottom to the top of the *Tonnetz* as seen in Figure 6. The cycles should always wrap around so that they travel horizontally across the *Tonnetz* and should never be expressed as travelling vertically.

Figure 5a V^7 in C major on the just *Tonnetz*

```
5   8   Y--2--5
        |V7__/
        |_/
1   4--7   X
    |I/
    |/
9   0   3   6
```

Figure 5b C major ii^7 on the *Tonnetz*

```
6   9   0   3   6   9--0   3   6
             ii7 /|  /
                / |/
2   5   8   Y--2--5   8   Y
            |V/
            |/
X   1   4--7   X   1   4
        |I/
        |/
6   9--0   3   6
   /|  /ii7
  / |/
2--5   8   Y
```

Figure 5c ii^{ø7} on the *Tonnetz*

```
6   9   0   3   6   9   0   3   6
               iiø7__/|
                 _/   |
2   5   8   Y--2--5--8   Y
            |V/
            |/
X   1   4--7   X   1   4
        |I/
        |/
6   9   0   3   6
    __/|
  _/    |iiø7
2--5--8   Y
```

Figure 5d vii^{o7} on the *Tonnetz*

```
5   8   Y--2   5   8


1   4--7   X   1   4
    |I/
    |/
9   0   3   6   9   3


5--8   Y   2   5   8
```

Figure 5e vii^{o7} on the *Tonnetz*

```
5   8   Y--2--5--8
            vii°7

1   4--7   X   1   4
    |I/
    |/
9   0   3   6   9   0



5   8   Y   2   5   8
```

Figure 5f The German augmented-sixth chord on the *Tonnetz*

```
6   9   0   3   6_  9   0--3   6
         Ger.#6 \_   |  /
                  \_|/
2   5   8   Y--2   5   8   Y
            |V/
            |/
X   1   4--7   X   1   4
        |I/
        |/
6_  9   0--3   6
  \_   |  /
    \_|/
2   5   8   Y
```

Figure 5g The French augmented-sixth chord. Notice that the contiguous pitches 9 and 5 are not part of the sonority:

```
6   9   0   3   6_  9  _0   3   6
        Fr.#6 |  \_/  |
              |_/  \_|
2   5   8   Y--2   5   8   Y
            |V/
            |/
X   1   4--7   X   1   4
        |I/
        |/
6_  9  _0   3   6
|  \_/  |
|_/  \_|
2   5   8   Y
```

These progressions by equal divisions of the octave will also require syntonic-comma corrections each time they cycle once around the tube. This comma shift must be imposed on an entire chord, as opposed to one or two members of a chord in the case of diatonic progressions. Realistically, such drastic comma corrections should not be imposed upon these progressions in performances that approximate just intonation. In fact, the path by which an enharmonically equivalent key area is achieved may well distinguish it perceptually from other instances of the same diatonic spelling. It may thus be useful for the analyst to distinguish between the key area three major thirds below tonic and the key area four minor thirds above tonic.[22] Even though using the tube *Tonnetz* model will rectify all equivalent spellings to one location, it will nevertheless show the path by which an enharmonically equivalent chord or key area is achieved.

The tube wrapping therefore allows most typical enharmonic cycles, after their respective number of syntonic-comma corrections, to slip by the same comma (648:625, or the major diesis). In the just-intonation *Tonnetz* model, this tuning difference between enharmonically equivalent pitches is a sort of corrected Pythagorean comma, as the Pythagorean comma, 531441/524288, multiplied by four syntonic commas, $(80/81)^4 = 40960000/43046721$, is equal to 625/648. As interval cycles other than fifths, however, shift by the same comma after receiving different numbers of syntonic-comma corrections, the more descriptive term "enharmonic comma" will thus be used.

As seen in Figure 6, the minor-second cycle travels enharmonically much faster than the other cycles, since the 15:16 ratio always functions as a diatonic step. Because the progression

[22] Lerdahl (1994) uses the *Tonnetz* to show "narrative paths" among equivalent and different tonal locations from a dramatic perspective (as opposed to simply enharmonically equivalent and different locations) in Wagner's *Parsifal*. Harrison (2000, 35-8) debates the rectification of all enharmonically equivalent locations on the *Tonnetz* and, along the way, (1994, 206-9, and 2000, 23-4) discusses the conceptual importance of distinguishing between syntonic-comma-related key areas in an equally tempered *Tonnetz* model of the last movement of Mahler's Second Symphony.

will travel twelve diatonic steps before returning to an enharmonically equivalent pitch class, the cycle will slip by five (= 12 - 7) 648:625 commas (after eight 81:80 corrections). No interval cycles other than those shown in Figure 6 will typically result when charting progressions of tertian chords on the *Tonnetz,* as root motion by diatonic intervals will be preferred. (Explicit enharmonic spelling strictures are given in part three.) If root motion by chromatic intervals is used, the discrepancy between chromatic and diatonic steps will cause the cycles to move much faster enharmonically. This is demonstrated in Figure 7 by the augmented-unison cycle, where the interval never functions as a diatonic step and will always remain on the same diatonic degree. Interval cycles involving other chromatic intervals exhibit similar characteristics on the just-intonation *Tonnetz*.

The representation on a flat sheet of paper of complicated diatonic and chromatic progressions constantly jumping from the bottom to the top of the *Tonnetz*, however, would not be elegant. This problem can be approached through a distinction in analytic technique between diatonic and chromatic progressions. Most diatonic progressions that begin and end on tonic (or some other contextually stable harmony) can be heard as prolongations at some analytic level. An enharmonic progression may also represent a prolongation of some harmony, but I value the use of the *Tonnetz* to show the potentially different aural effect of this type of progression: a path to a chromatically different chord that is only then retrospectively reinterpreted as the original harmony itself. Riemann ([1877] 1971, 119) offers a similar description of the effect of enharmonicism in Wagner's music:

> Wagner has his reasons for sometimes descending into ♭♭s or ascending into ♯♯s. He also knows that the greater part of his audience can follow him down below and over above these tortuous paths. He knows that a not inconsiderable few of these effects are to

> be ascribed to orthography, and that he can even communicate them to untrained listeners, without their knowing it.[23]

Now that any diatonic progression can be shown on the *Tonnetz* as motion around a single tonic (possibly wrapping around the tube), diatonic progressions are potentially extraneous to exploration of enharmonic motion and would only obfuscate a graph. The stability afforded by syntonic-comma corrections upon diatonic progressions allows the representation of enharmonic progressions to remain the same regardless of the inclusion or exclusion of diatonic progressions.[24] When the analyst reduces out diatonic prolongations, the chromatic progressions can now be shown as an interval cycle as seen in Figure 6, or as a combination of several different interval progressions.

[23] Hyer 1995, 106.
[24] A technique for exploring prolongations and succinctly showing the melodically fluent voice leading connections that create diatonic (as well as enharmonic) prolongations is given in Jones 2002.

Figure 6 Interval cycles on the just-intonation *Tonnetz*.

Augmented-fourth / diminished-fifth (ic 6) cycle:

```
5   8   Y   2   5   8   Y   2


1   4---7   X---1   4---7   X
    |  /    |  /    |  /
    | /  →  | /  →  | /
    |/      |/      |/
9   0   3   6   9   0   3   6


5   8   Y   2   5   8   Y   2


1   4   7   X   1   4   7   X


9   0   3   6   9   0   3   6
```

Perfect-fourth / perfect-fifth (ic 5) cycle:

```
  5   8   Y   2---5---8---4---2
              |  /|  /|  /|  /
            ←| /←| /←| /←| /
           ↓ |/↓ |/↓ |/↓ |/
  1   4---7---X---1---4---7   X
      |  /|  /|  /|  /|  /
    ←| /←| /←| /←| / | /
   ↓ |/↓ |/↓ |/↓ |/  |/
  9---0---3---6---9   0   3   6
  |  /|  /|  /|  /
←| /←| /←| /←| /
↓ |/↓ |/↓ |/↓ |/
  5   8   Y   2   5   8   Y   2
```

Major-third / minor-sixth (ic 4) cycle:

```
5   8   Y   2   5   8---Y   2
                    |  /
                    | /
                    |/↓
1   4---7   X   1   4---7   X
    |  /            |  /
    | /             | /
    |/↓             |/↓
9   0---3   6   9   0   3   6
    |  /
    | /
    |/↓
5   8   Y   2   5   8   Y   2
```

Minor-third / major-sixth (ic 3) cycle:

```
1   4---7---X---1---4---7   X
    |  /|  /|  /|  /|  /
    | /←| /←| /←| /←| /
    |/  |/  |/  |/  |/
9   0   3   6   9   0   3   6


5   8   Y   2   5   8   Y   2


1   4   7   X   1   4   7   X
```

Major-second / minor-seventh (ic 2) cycle:

```
                ↑       ↑
 5   8   Y   2---5   8---Y   2
             |  /    |  /
         →→→→| / →→→→| /
        ↑    |/ ↑    |/
 1   4---7   X---1   4---7   X
     |  /    |  /    |  /
     | / →→→→| / →→→→| /
     |/ ↑    |/ ↑    |/
 9   0---3   6---9   0   3   6
     |  /    |  /
 →→→→| / →→→→| /
↑    |/ ↑    |/
 5   8   Y   2   5   8   Y   2
```

Figure 6, continued

Minor-second / major-seventh (ic 1) cycle:

```
                ↑           ↑       ↑           ↑       ↑           ↑       ↑           ↑
  0   3   6   9 ↑ 0   3   6---9   0 ↑ 3   6   9---0   3 ↑ 6   9   0---3   6 ↑ 9   0   3---6
                ↑         |  /      ↑         |  /      ↑         |  /      ↑         |  /
                ↑        →| /       ↑        →| /       ↑        →| /       ↑        →| /
                ↑       ↑ |/        ↑       ↑ |/        ↑       ↑ |/        ↑       ↑ |/
  8   Y   2   5---8   Y ↑ 2   5   8---Y   2 ↑ 5   8   Y---2   5 ↑ 8   Y   2---5   8 ↑ Y   2
              |  /      ↑         |  /      ↑         |  /      ↑         |  /      ↑
             →| /       ↑        →| /       ↑        →| /       ↑        →| /       ↑
            ↑ |/        ↑       ↑ |/        ↑       ↑ |/        ↑       ↑ |/        ↑
  4---7   X ↑ 1   4   7---X   1 ↑ 4   7   X---1   4 ↑ 7   X   1---4   7 ↑ X   1   4---7   X
  |  /      ↑         |  /      ↑         |  /      ↑         |  /      ↑         |  /
 →| /       ↑        →| /       ↑        →| /       ↑        →| /       ↑        →| /
↑ |/        ↑       ↑ |/        ↑       ↑ |/        ↑       ↑ |/        ↑       ↑ |/
  0   3   6   9   0   3   6   9   0   3   6   9   0   3   6   9   0   3   6   9   0   3   6
```

Figure 7 Augmented-unison / diminished-octave cycle:

```
1   4---7   X   1   4   7   X   1---4   7   X   1   4   7   X---1   4   7   X   1   4   7---X   1   4   7   X   1   4---7
    |  /                        |  /                        |  /                        |  /                        |  /
    | / →→                      | / →→                      | / →→                      | / →→                      | /
    |/    ↓                     |/    ↓                     |/    ↓                     |/    ↓                     |/
9   0   3---6   9   0   3   6   9   0---3   6   9   0   3   6   9---0   3   6   9   0   3   6---9   0   3   6   9   0   3
        |  /                        |  /                        |  /                        |  /
        | / →→                      | / →→                      | / →→                      | / →→
        |/    ↓                     |/    ↓                     |/    ↓                     |/    ↓
5   8   Y   2---5   8   Y   2   5   8   Y---2   5   8   Y   2   5   8---Y   2   5   8   Y   2   5---8   Y   2   5   8   Y
            |  /                        |  /                        |  /                        |  /
            | / →→                      | / →→                      | / →→                      | / →→
            |/    ↓                     |/    ↓                     |/    ↓                     |/    ↓
1   4   7   X   1   4   7   X   1   4   7   X   1   4   7   X   1   4   7   X   1   4   7   X   1   4   7   X   1   4   7
```

Figure 8 Comparing the geometry of the neo-Riemannian *Tonnetz* in equal temperament with the 5-limit just-intonation *Tonnetz*

```
12-Tone Equal Temperament

6   9   0   3   6

2   5   8   Y   2

X   1   4   7   X

6   9   0   3   6
```

```
Tempering major thirds
  ____________
 /            /\
 6  9  0  3  6||
 \____________\/
```

```
Tempering minor thirds
   _________
  /         \
  |    _     |
  |   \_/    |
  \          /
   \   6    /
    \______/
```

```
5-Limit Just Intonation

6   9   0   3   6   9   0   3   6

2   5   8   Y   2   5   8

X   1   4   7   X   1

6   9   0   3   6
```

```
Correcting syntonic comma
               _______________
             /\               \
            /  \               \
   ______  /____\               \
  \6  9  0  3   6__9__0__3__6\
   \             \    /
    \             \  /
     \____________\/
```

```
Correcting enharmonic shifts
   _________
  /         \
  |    _     |
  |   \_/    |
  \          /
   \   6    /
    \______/
```

III. Analysis of Enharmonicism on the Just Intonation *Tonnetz*

A number of analytical problems will now be explored through several typical examples of enharmonic progressions as shown on the *Tonnetz*. All of these progressions will be drawn from the Two Rhapsodies, Op. 79, by Brahms. The Rhapsodies are among Brahms' most chromatic works for keyboard, and the tonally ambiguous openings of both of them have been discussed by several scholars.[25] Indeed, neither work establishes its key in the first few measures through use of a traditional resolution of a functional dominant harmony to a stable tonic chord. Although the second Rhapsody in G minor eventually features more traditional progressions in the tonic major key beginning in measure 9, the first Rhapsody in B minor, as seen in Example 1a, features no truly stable tonic sonority anywhere in its first tonal area before it begins the process of modulating to the dominant (in m. 4). The nature of Brahms' use of the prolonged F♯ major chords in the first four measures (in effect prolonging a dominant seventh chord), however, clearly implies dominant function in B minor. I will analyze the first 16 measures of this Rhapsody here, but I do not intend for my analysis to offer insights into this harmonic ambiguity. My goal is instead to codify Brahms' use of a chromatic progression that travels to an enharmonically equivalent, but tonally different chord from the F♯ opening of the piece.

The first 16 measures of the first Rhapsody are given in Example 1a. A harmonic analysis is provided, and the chords are numbered 1 - 31. The reason for performing an analysis of the implied harmonies in the unaccompanied octave melody in mm. 13 - 15 will be discussed below in conjunction with Example 2. The harmonic content of these 31 chords is abstracted

[25] An analysis of the opening of the first Rhapsody is found in Schenker [1906] 1954, ex. 24, pp. 33-34. Analyses of the opening of the second Rhapsody are found in Schenker [1906] 1954, ex. 28, pp. 35-37; Schoenberg 1954, ex. 164, pp. 175-177; and Jonas, appendix to Schenker [1906] 1954, ex. A5, p. 345. Greenberg (1976) compares these analyses, and a response by Charles Smith follows the article.

into the prolongational graph given in Example 1b. The pitches on the graph are not intended to imply a particular register. Instead, all pitches given are representatives of their pitch class. All alternative inversions of the chords and octave transpositions of the pitches would thus be equivalent. The prolongations that are shown in the example reflect one possible reading of this passage. Passing and/or neighboring motions in the diatonic pitch-class voice leading connect all members of the two outer chords that are slurred together on the graph.[26] Chords 19 - 31 are respelled enharmonically with respect to the notation in the score. This respelling holds common tones to the same diatonic scale step between the F minor chord (18) and the C♯^{7}/D♭^{7} chord (19). A second analytical level is provided, which reduces out the diatonic prolongations shown in the first level and shows the main chromatic motion of the excerpt.

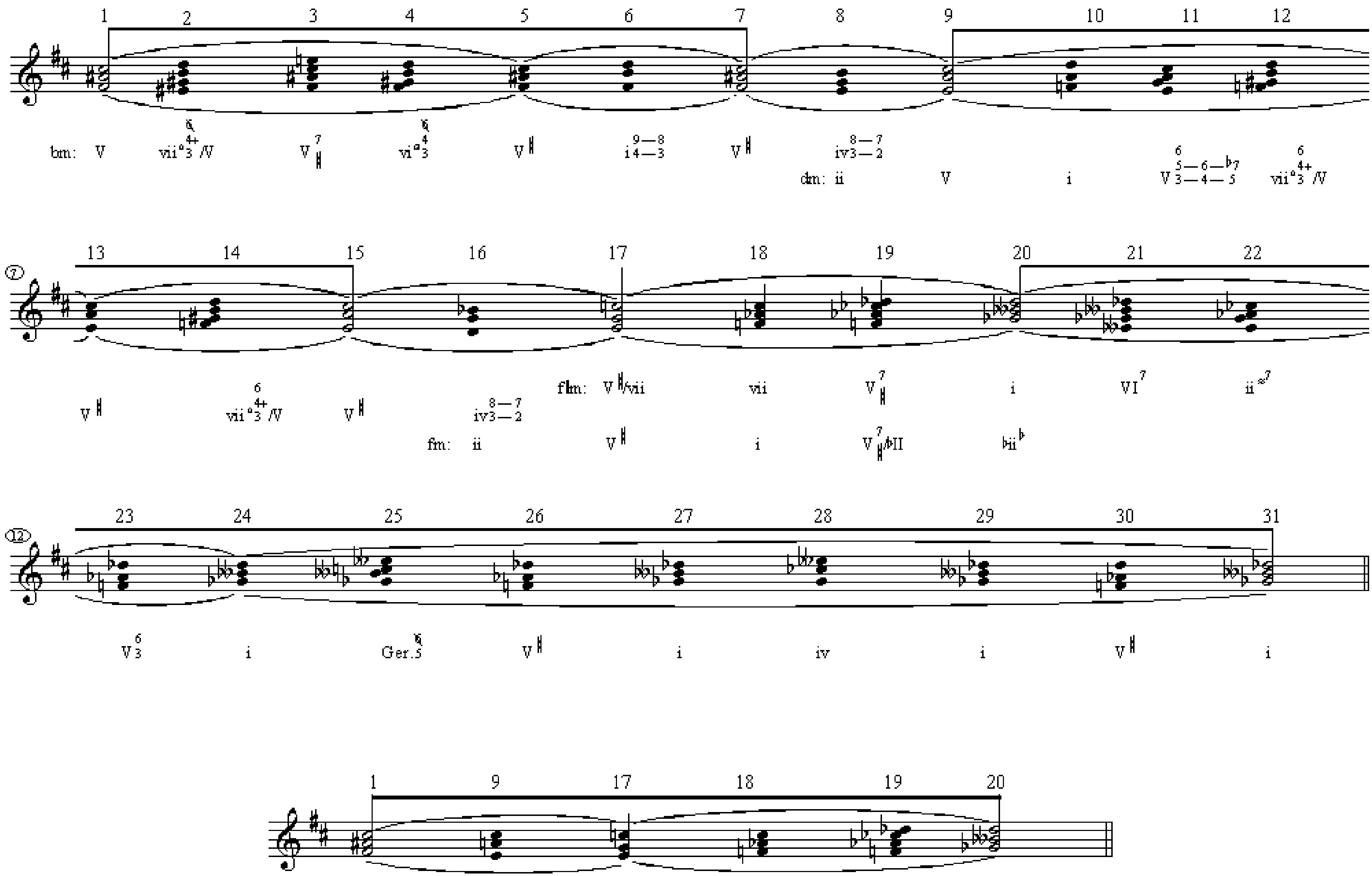
Using the method of *Tonnetz* representation outlined above, this chord succession can be graphed as shown in Example 1c. Every pair of adjacent chords features at least one common tone and a root relationship expressed as a diatonic interval. There is an incomplete minor third cycle between chords 1, 9, and 17. After moving two chords along the minor third cycle, the C major chord (17) moves to an F minor chord (as a V to i progression) and then holds two common tones into the V^7 of F♯/G♭ minor (19). The progression, with one 81:80 comma correction, ends one 648:625 enharmonic comma away from the original F♯ (in other words, on G♭ minor). The enharmonic motion would be unchanged in a *Tonnetz* representation that includes the removed prolongational progressions. With such a detailed graph, however, the means by which the music moves away from the first chord would not be nearly as clear visually.

---

[26] Jones (2002) defines this property as "pervasive fluency," after Schenker's "melodic fluency." According to Jones, the passing and neighboring motions offer the analyst a way of listening to a passage in order to hear it as being a prolongation.

Example 1a Brahms, Rhapsody in B Minor, Op. 79, No. 1, m. 1–16

Example 1b Brahms, Rhapsody in B Minor, Op. 79, No. 1, m. 1–16



Example 1c Brahms, Rhapsody in B minor, Op. 79, No. 1, mm. 1 - 16 on the *Tonnetz*

```
4   7   X---1---4---7   X   1
        |#1/|#9/|17/       /|
        | / | / | /       / |
        |/  |/  |/       /20|
0   3   6   9   0   3   6---9
               /|
              / |
             /18|
8   Y   2   5---8---Y   2   5
            | #19__/
            | __/
            |/
4   7   X   1   4   7   X   1
```

Example 2a shows a passage from the recapitulation of the B minor Rhapsody. The unaccompanied octave melody is seen again in this section (this time transposed to the tonic key), and it is immediately presented a second time, now transposed up a minor second and completely harmonized. After listening to the piece a number of times, a trained listener is likely to begin to hear the unaccompanied version of the melody similarly to the accompanied version in terms of harmony. Hence, I do not hesitate to provide a harmonic analysis for the octave-doubled monophonic passage.[27] A prolongational graph is provided in Example 2b. In order to express half-step motion as minor seconds, chords 10 - 14 are spelled differently from the more practical spellings found in the score.

The graphing of this progression on the *Tonnetz*, however, presents certain problems. Holding all common tones among chords 11 - 14 would cause the vii°^{7} (chord 13) to be misspelled with regard to its resolution. This is because a minor third in the previous chord (12) between pc 7 and pc 10 would have to be respelled as an augmented second for the vii°^{7} chord to resolve properly. In order to resolve this dilemma, guidelines for locating ambiguous or enharmonically problematic harmonies on the *Tonnetz* would be beneficial at this point.

[27] For more on justifying the use of implied tones in analysis, see Rothstein 1991.

Enharmonic spelling strictures:

1. Diatonic-spelling stricture: Decide the spelling of the chordal root based on diatonic-interval root relationships among chords, and, in the case of leading-tone diminished-seventh chords, minor-second leading-tone motion to the root of the resolution chord. Within each chord, preserve tertian interval spellings from the chord's root.

2. Common-tone stricture: Do not enharmonically respell perfect-unison common tones between chords, unless the tertian spelling of a chord is thus violated by interval conflicts (m3/A2, d5/A4, m7/A6). This typically will only occur when the composer has enharmonically reinterpreted French augmented-sixth chords, German (or Italian) augmented-sixth chords and dominant-seventh chords, or diminished-seventh chords. Always respell the least number of unison common tones possible.

3. Non-traditional chord resolutions: Deciding the enharmonic spelling of the root of certain chords such as diminished-seventh chords, and identifying the root of symmetrical chords (augmented triad and French sixth) may be problematic when the chords are not used traditionally, or when diatonic root motion is in conflict with the holding of common tones. In this case, respelling unison common tones is forbidden, both when held from the previous chord and when held into the following chord. If there are no common tones, or there is still more than one spelling possibility, maximize minor and major second pitch-class motions among all chords involved.

4. Augmented-unison stricture: Use augmented unisons only to preserve the internal tertian spellings of adjacent chords once their root spellings have been decided using Rules 1 - 3.

5. Tritone-relation stricture: Abrupt tritone modulations and tritone-related chord successions may still be ambiguous (♯IV/♭V). In this case, the analyst must look at the relationship that ♯IV/♭V bears to the surrounding, more closely related chords.[28]

6. Non-chord-tone stricture: The use of non-chord tones can sometimes allow for violations of the diatonic-spelling stricture as long as the melodic pattern formed by the non-harmonic note is diatonic, or, in the case of chromatic passing tones, postpones diatonic half-step motion until the end of the chromatic line. However, the diatonic-spelling stricture cannot typically be overruled if the non-chord tone forms a separate, but analyzable, harmony when combined with the sounding chord tones.

---

[28] Brown, Dempster, and Headlam (1997) deal with this issue in a similar manner in the attempt to establish a Schenkerian definition of the outer limit of tonality.

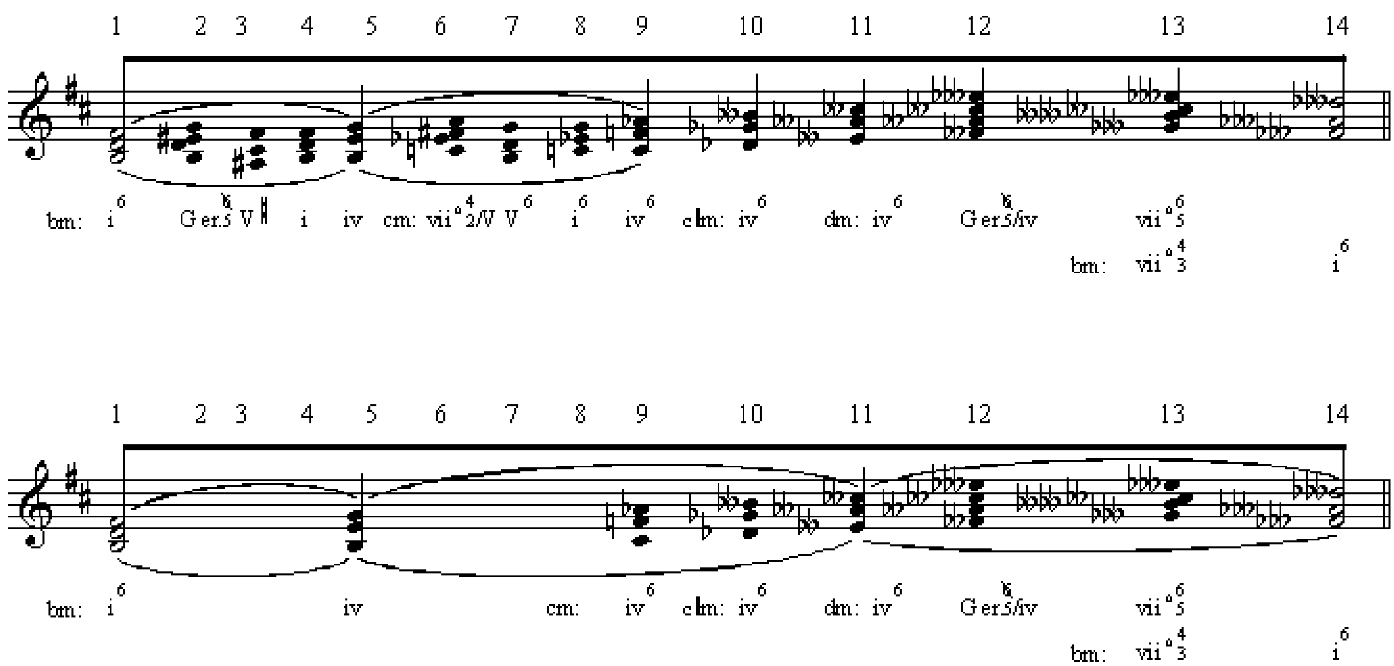
Because diatonic root motions, functional resolutions of leading tones, and correct tertian spellings are essential in order for the *Tonnetz* to represent tonal harmony (Rule 1), it is thus necessary to create the proper leading tone resolution from the root of the vii°^{7} (chord 13) to the root of chord 14. Because the minor-third / augmented-second conflict discovered above results from the enharmonic reinterpretation of a diminished-seventh chord, the common-tone stricture (Rule 2) offers the provision to respell one of the common tones between chord 12 and chord 13. In this case, the non-traditional resolution stricture (Rule 3) will decide the enharmonic spelling of chord 13 with respect to chord 12. As seen in Example 2c, If the tritone in chord 12 (pc 7 to pc 1) is to be spelled as it will eventually resolve, the chord must be spelled as a German augmented-sixth chord. This necessitates that pc 10 in chord 13 be respelled, in order to obtain the augmented second between pcs 7 and 10 without tampering with the spelling of this tritone. Pc 3 in chord 12 would have to resolve to pc 4 by augmented unison (25:24).

Example 2d shows a second option. If instead we wish to respell pc 7 in chord 13 to form the augmented second between pcs 7 and 10, then chord 12 should be spelled as a dominant seventh chord to hold the maximum number of common tones. Respelling from a German-sixth chord to a dominant-seventh chord is, of course, not a violation of the chord's tertian spelling, and pc 3 in chord 12 will now resolve to pc 4 by minor second (16:15). Because both options hold the same number of common tones, Rule 3 favors the spelling that maximizes minor second motion (pc 3 in chord 12 to pc 4 in chord 13). This option is still preferable even though, in this case, respelling pc 7 would cause a shift of one additional enharmonic comma away from the original B minor.

Example 2e shows the entire passage on the *Tonnetz*. The progression is primarily a minor second cycle, interrupted after 4 steps by a much quicker motion to back to an enharmonically equivalent B-minor sonority. The *Tonnetz* representation clearly shows both the method by which the music moves through tonal space and the tonal distance travelled. The progression from the work's recapitulation seen here thus travels much further afield enharmonically than did the progression from the exposition.

Example 2a Brahms, Rhapsody in B minor, mm. 79–86

Example 2b



Example 2c Brahms, Rhapsody in B minor, Recapitulation, chords 11 - 14

```
5   8   Y   2   5   8   Y   2   5
                           /|
                          / |
                 13      /11|
1   4   7   X===1===4===7---X   1
                 \__    |12/
                    \__ | /
                       \|/
9   0   3   6   9   0   3   6   9
           /|
          / |
         /14|
5   8   Y---2   5   8   Y   2   5
```

Example 2d Brahms, Rhapsody in B minor, chords 11 - 14, Recapitulation, second interpretation

```
5   8   Y   2   5   8   Y   2   5   8   Y
                           /|
                          / |
                         /11|    13
1   4   7   X   1   4   7---X===1---4---7
                        |12  __/
                        | __/
                        |/
9   0   3   6   9   0   3   6   9   0   3
                           /|
                          / |
                         /14|
5   8   Y   2   5   8   Y---2   5   8   Y
```

Example 2e. Brahms, Rhapsody in B minor, Recapitulation, mm. 78 - 86

```
4   7   X   1   4   7   X   1   4   7   X
                           /|
                          / |
                         /10|
0   3   6   9   0   3   6---9   0   3   6
       /|      /|                      /|
      / |     / |                     / |
     /#1|    /#9|                    /14|
8   Y---2   5---8   Y   2   5   8   Y---2
   /|                  /|
  / |                 / |
 /#5|                /11|
4---7   X   1   4   7---X   1   4   7   X
```

The final example of typical enharmonic motion on the just-intonation *Tonnetz* is a longer passage encompassing most of the development section of the second Rhapsody in G minor. A harmonic analysis is given in Example 3a, and a harmonic reduction showing prolongations is given in Example 3b. For clarity of notation on the first level graph, enharmonic spellings that preserve common tones and diatonic stepwise motion are postponed until the second level prolongational graph. The first chord, D minor, is representative of the second key area in the exposition. Chords 14 - 32 are essentially a sequential repetition of chords 2 - 13, transposed down a major third and featuring a more extensive dominant prolongation between chords 21 and 32. The second and third levels insert a G minor chord as chord 0, to represent the first tonal area of the exposition for comparison with chord 46. (There are no enharmonic progressions in the exposition.) Example 3c presents chords 0, 1, 9, 21, 33, and 46 on the *Tonnetz*. This *Tonnetz* representation shows that the primary harmonic motion of this example is a descending major third cycle between chords 1, 9, and 21. Chord 21 resolves authentically to chord 33, which then moves through a quick harmonic sequence (not shown) to chord 46, one enharmonic comma away from chord 0. This method of *Tonnetz* representation thus functions equally well for longer passages, provided one is willing to view the music from a prolongational perspective.[29]

[29] A background view of the piece, where even an enharmonic prolongation would be reduced out, would have to rectify the 645:625 comma and thus show no enharmonic motion on the *Tonnetz*. This can be accomplished by wrapping the tube into a torus, as seen at the bottom of Figure 8. As with other progressions, the process by which this must occur is comma correction. With these quite audible *enharmonic comma* shifts, however, the "seams" of the *Tonnetz* will begin to show.

Example 3a Brahms, Rhapsody in G minor, Op. 79, No. 2, Development


49
27
28
29
30
31
V♭9 7
vii°4 2/V
V♭9 7
I6 4
vii°♭7/V
53
32
33
34
35
36
37
V
i
VI7
ii4 3
v6 5
i4 3
58
38
39
40
41
42
43
iv6 5
gm:
vi6 5
VI4 3
ii°6 5
V♭9 4
vi6 5
VI4 3
62
44
45
46
mg
ii°6 5
V♭9 4
i

Example 3b Brahms, Rhapsody in G minor, Op. 79, No. 2, Development

Example 3c Brahms, Rhapsody in G minor, Op. 79, No. 2, Development.

```
4   7   X   1   4   7   X---1---4   7   X
                        | #21__/
                        | __/
                        |/
0   3   6   9   0   3   6   9   0   3   6
           /|          /|
          /v|         / |
         /#1|        /33|
8   Y   2---5---8   Y---2   5   8   Y   2
       /| #9 __/       /|
      /i| __/         / |
     /#0|/           /46|
4   7---X   1   4   7---X   1   4   7   X
```

IV. Treating Unusual Chord Resolutions and Functional Ambiguity

Whenever enharmonic ambiguity arose in the Brahms Rhapsody examples, the chord succession in question always contained at least one typical chord resolution to rely upon in order to aid in deciding on the spelling. The unusual chord resolutions present at the opening of the Prelude to Wagner's *Tristan und Isolde* will provide a preliminary example of chord successions involving exclusively atypical resolutions of tertian chords that normally resolve functionally.[30] A piano reduction of the excerpt is given in Example 4a. Typical functional resolutions begin with the B dominant seventh chord (6) resolving to the E dominant ninth chord (7). Example 4b shows a harmonic reduction of the first eight chords.

It will be essential here to determine how to spell the first chord. When spelled as an F half-diminished seventh chord, the analyst would be emphasizing its attributes that suggest a French augmented-sixth chord.[31] When spelled as an E♯ half-diminished seventh chord, the analyst would be emphasizing its "altered dominant" quality by holding the common tones into the actual dominant seventh immediately following it.[32] Although there may well be some functional ambiguity here, the two interpretations are quite distinct when taken from a just-intonation perspective. The reason for the functional ambiguity here is a conflict between the preference for diatonic root motion (Rule 1) and the preference for holding common tones (Rule 2). The non-traditional resolution stricture (Rule 3) indicates that in cases where these two rules are in conflict, common tone respelling is forbidden. This interpretation is given in Example 4c.

[30] This prelude has historically been a proving ground for the capability of tonal theories in interpreting chromatic harmony. Harrison (1994, 153-7) and Smith (1986) both offer a functional analysis of this passage. Lewin (1996) discusses the symmetry of these transformations in twelve-tone equal temperament, and Douthett and Steinbach (1998) build upon this analysis.

[31] Smith notes that this would be a somewhat atypical resolution of the French augmented-sixth chord.

[32] Smith (1986) supports this argument as well, allowing for both possibilities in a "functionally extravagant" reading.

A more surface-level reading of the passage, adding the chromatic passing tones A and A♯ into the transformation, will contribute to the decision of the more preferable interpretation in just intonation.

Example 4d shows the two passing sonorities between chords 1 and 2, given as chords 1a and 2a. Both chords are French augmented-sixth chords, and, according to diatonic spelling stricture (Rule 1) must be tuned internally as shown. Because chord 1a shares three common tones with chord 1 (E♯/F half-diminished), the non-traditional resolution stricture (Rule 3) dictates that chord 1a must be represented as a B/C♭ French sixth (rather than an E♯/F French sixth).[33] Likewise, because chord 2a shares three common tones with chord 2 (E/F♭ dominant seventh), it would be preferable to spell chord 2a as a A♯/B♭ French sixth (rather than an E/F♭ French sixth). As there are no common tones between chords 1a and 2a, the non-traditional resolution stricture (Rule 3) will opt for the maximization of minor second motion. Furthermore, Rule 1 favors root motion by diatonic intervals over root motion by chromatic intervals. Thus we will prefer the progression to be spelled as F half-diminished seventh, C♭ French sixth, B♭ French sixth, E dominant seventh. Intuitively, the more diatonic resolution of chord 1a to chord 2a makes sense because with this spelling chord 2a gives the expected minor-second resolution of the members of the augmented sixth interval in chord 1a (pc 11 and pc 9). A surface-level view of the voice leading thus reinforces the interpretation that prefers diatonic-interval root motion (Rule 1) between chords 1 and 2 rather than common-tone retention (Rule 2).

As seen in Example 4e, Chord 2 must preserve diatonic interval root motion into chord 3, which in this case also preserves all common tones (no rule conflicts). This also causes an

[33] "B/C♭ French sixth" refers to the diatonic representation of the set class 4–25 (0268) that often uses pc 11 as its bass note and always spells the interval between pc 11 and pc 9 as an augmented sixth or diminished third.

enharmonic comma shift, however, as chords 3 and 4 bear the same relationship to one another as the first two chords. Chords 4 and 5, shown in Example 4f, also preserve both diatonic interval root motion (Rule 1) and unison common tones.

There is a different relationship between chords 5 - 6, shown in Example 2g, than between chords 1 - 2 and 3 - 4. In this case, there is a conflict between the diatonic root motion interpretation (Rule 1) and, as there are no common tones, the interpretation that favors diatonic half step resolutions (Rule 3). Choosing diatonic root motion would require the interval between pc 2 and pc 11 to be spelled as a minor third (which is not the case in Example 2g). Example 4g shows the option which maximizes minor second resolutions. This sacrifices the diatonic resolution of the descending half step (pc 0 to 11) in favor of the three ascending half steps. As with chords 1 to 2 and 3 to 4, however, chord 5 undergoes quite significant surface-level changes before resolving to chord 6, which will inform our just-intonation interpretation and resolve the conflict between Rule 1 and Rule 3. These changes can be seen in Example 4h, where chords 5a, 5b and 6a represent the intermediate verticalities.

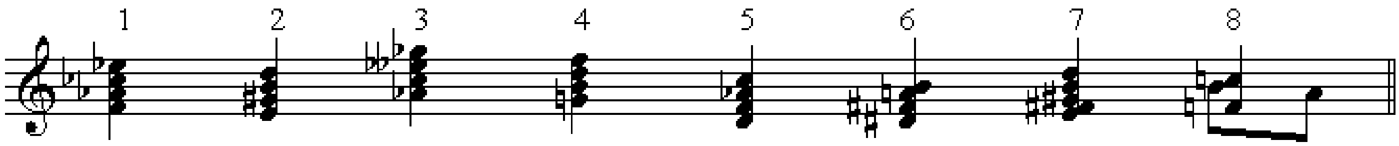
First, in chord 5a, pc 2 moves up by a half step to pc 3, creating an F minor seventh chord (0358). Next, pc 3 and pc 5 converge on pc 4 in chord 5b, leaving an augmented triad (048) sounding. This chord's tendencies with regard to its resolution into chord 6a (Y359) are quite different from chords 5 and 6, as all three pitch classes can resolve by minor second to members of chord 6a without violating the tertian spelling of either chord. In order to spell chord 5b so that it maximizes minor second resolutions into chord 6a, pc 0 must be the root, and pc 8 must thus be respelled. This respelling, however, is not provided for by Rule 2, since this case does not constitute a conflict of functionally consonant and dissonant intervals. Diminished fourths are indeed unstable relative to major thirds, but they are not required to resolve to minor thirds.

Unlike the augmented second in diminished seventh chords, it is far less jarring to the ear to resolve the diminished fourth in an augmented triad to a major third than to respell a common tone. (This may have something to do with the fact that the unstable interval in the dim.-7th chord involves the chord's seventh, which is not the case with the augmented triad.) The non-traditional resolution stricture, then, will decide which member of chord 5b is its root. Example 4h shows the spelling that holds all of the common tones and maximizes minor second motions. The progression from chord 5 to chord 6 therefore does not drift by an enharmonic comma as it would have in Example 4g, and thus supports the preference for diatonic root motion over maximization of minor second motion set up in the hierarchy of enharmonic spelling rules.

This last example has shown that a close reading of the voice leading at the surface level is necessary for an unambiguous *Tonnetz* representation of chords in a progression. The only voice leading "parsimony" required for progressions to be able to be represented transformationally in tonal space is the possibility of finding local diatonic stepwise and common-tone voice leading. A consistent just-intonation reading of highly chromatic music is thus possible using the spelling strictures given above, as long as all of the chords either are tertian or follow traditional functional resolutions. The value of using such a just-intonation model for music typically rendered in tempered systems lies in its ability to offer insights into music through spatial metaphors tonal motion. The recognition of how key areas are established, approached, left, and reattained, and the qualities they can take on based on this context, can play a role in the interpretive decisions made by performing musicians, regardless of whether an underpinning of just intonation in musical perception is a reality. A just-intonation perspective is merely a way of approaching music that offers specific interpretive rewards that are lost when one's musical imagination is restricted to equal temperament's democracy of tones. Any analytical techniques that work under the assumption of equal temperament can, of course, be used alongside this method and also contribute to the richness of one's understanding of a composition. The just-intonation *Tonnetz* is therefore a valuable theoretical tool for understanding both music that involves enharmonic shifts and highly chromatic music that transcends traditional tonal function.

Example 4a Wagner, *Tristan und Isolde*, Prelude, mm. 1 - 17

Example 4b   Wagner, *Tristan und Isolde*, Prelude



Example 4c Wagner, *Tristan und Isolde*, chords 1 and 2, common tone interpretation.

```
0   3   6   9   0   3   6   9   0
               ____/|
              / #1  |
8   Y   2   5--8--Y--2   5   8
                |  ____/
                |/#2
4   7   X   1   4   7   X   1   4
```

Example 4d Voice leading from chords 1 to 2

```
0   3   6   9_  0  _3   6   9   0
         #1a|  \_/  |
            |_/   \_|
8_  Y  _2   5   8   Y   2   5   8
|  \_/  |
|_/   \_|#2a
4   7   X   1   4   7   X   1   4
```

Example 4e Chords 2 and 3

```
0   3   6   9   0   3   6   9   0
   ____/|
  / #3  |
8--Y--2   5   8   Y   2   5   8
|  ____/
|/#2
4   7   X   1   4   7   X   1   4
```

Example 4f Chords 4 and 5

```
0   3   6   9   0   3   6   9   0
           ____/|
          / #5  |
8   Y--2--5--8   Y   2   5   8
    |  ____/
    |/#4
4   7   X   1   4   7   X   1   4
```

Example 4g Chords 5 and 6, interpretation favoring minor second motion

```
0   3   6   9   0   3--6--9   0
           ____/|   |  ____/
          / #5  |   |/#6
8   Y   2--5--8   Y   2   5   8


4   7   X   1   4   7   X   1   4
```

Example 4h Voice leading from chords 5 to 6

```
8   Y   2   5   8   Y   2   5   8


4   7   X   1   4   7   X   1   4
                |
                |5b
0   3   6   9   0--3   6   9   0
           ____/| /5a
           /#5/ |/
8   Y   2--5--8   Y   2   5   8


4   7   X   1   4   7   X   1   4




8   Y   2   5   8   Y   2   5   8


4   7   X   1   4   7   X   1   4
                |
                |5b
0   3_  6  _9   0   3   6   9   0
  6a|  \_/  |   |
    |_/   \_|   |
8   Y   2   5   8   Y   2   5   8


4   7   X   1   4   7   X   1   4
```

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